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Distributional Methods in Saturation Theory[1]

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1. INTRODUCTION AND PRELIMINARIES

The points of departure of this note are a series of papers by Butzer [*7*], [*9*], [*10*] who introduced the Fourier transform method and developed the one-dimensional saturation theory, the papers of Nessel [*25*], [*26*] and Butzer and Nessel [*13*] in which the corresponding n-dimensional problems were treated, as well as a paper of the author and Nessel [*21*] where characterizations of generalized derivatives were established by means of some simple results of the theory of distributions. The essential aim of this paper is to solve the characterization problem for saturation classes as well as to determine the saturation classes themselves for functions of several variables by using distribution theoretical methods. These methods enable one to treat successfully several questions, in particular the connection between saturation classes and the spaces of Bessel potentials, the Sobolev spaces, as well as Lipschitz conditions and ordinary differentiability properties, and various types of special Besov spaces which also play an important role in the theory of partial differential equations (cf. e.g. [*23*]). In the two cases that are of most importance concerning applications (see Sections 3 and 4) we obtain a complete set of characterizations valid on $L^p(E^n)$, $1 < p < \infty$.

Since there is an essential difference in statement and method of proof between non-optimal approximation and optimal, i.e., the saturation-case, the subject of this paper may be regarded as a solution of an extremal case in a certain sense. Indeed, the theorems on non-optimal approximation for, e.g., the singular integral of Weierstrass on E^1 (cf. (1.4) for the definition) as established by Berens [*3*] and Taibleson [*32*], and the saturation theorems for these approximation processes (see, e.g., Butzer [*7*], [*9*] and Nessel [*25*], [*26*]) cannot be established by the same methods but are complementary to each

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[1] This paper contains the complete proofs of the results announced in part in a note under the title: Characterizations of Favard classes for functions of several variables, *Bull. Am. Math. Soc.* **74** (1968), 149–152. Some of these results have been presented in a lecture held on September 13, 1967 at the annual meeting of the German Mathematical Society at Karlsruhe. Some of these results are part of the author's doctoral dissertation written under the direction of Prof. Dr. P. L. Butzer at the Technological University of Aachen.

other. Correspondingly, the characterizations of Lipschitz conditions of the type

$$\|f(x+h)-f(x)\|_{L^p(E^n)} = O(|h|^\alpha),\ |h| \to 0, \qquad \text{for } 0<\alpha<1,$$

given by Taibleson [32], do not include the theorems of Hardy–Littlewood type which state that in case $\alpha = 1$, $1 < p < \infty$, and, e.g., $n = 1$, the Lipschitz condition is equivalent to the existence of $f'(x) \in L^p(E^1)$ (cf. Butzer [8]). From this point of view, optimal approximation as well as the characterization problem for saturation classes appear as extremal cases of non-optimal approximation, in the sense that $\alpha = 1$ may be regarded as an extremal index for the above Lipschitz space. It is to be noted that the methods of proof in the extremal case are the more difficult ones.

Let E^n denote the Euclidean n-space of real vectors

$$x = (x_1, x_2, \ldots, x_n), \qquad v = (v_1, v_2, \ldots, v_n),$$

with

$$(x, v) = \sum_{j=1}^{n} x_j v_j \quad \text{and} \quad |x| = (x, x)^{1/2}.$$

In the sequel, x, u, v, h will always denote n-dimensional variables, whereas τ, s, t are elements of E^1.

$L^p(E^n)$, $1 \leqslant p < \infty$, denotes the space of all complex-valued functions f having norm

$$\|f(x)\|_p = \left(\int_{E^n} |f(x)|^p\, dx\right)^{1/p} < \infty,$$

and $M(E^n)$ denotes the space of bounded measures μ on E^n with norm $\|\mu\|_M \equiv \int_{E^n} |d\mu(x)|$.

If $1 \leqslant p \leqslant 2$, the ordinary Fourier transform of $f(x) \in L^p(E^n)$ is defined by

$$\mathfrak{F}[f](v) \equiv f^\wedge(v) = \begin{cases} (2\pi)^{-n/2} \displaystyle\int_{E^n} e^{-i(v,x)} f(x)\, dx, & p = 1 \\ \underset{R\to\infty}{\overset{(p')}{\text{l.i.m.}}}\ (2\pi)^{-n/2} \displaystyle\int_{|x|\leqslant R} e^{-i(v,x)} f(x)\, dx, & 1 < p \leqslant 2 \end{cases} \tag{1.1}$$

where $\text{l.i.m.}^{(p')}$ denotes the limit in $L^{p'}$-norm, $(1/p) + (1/p') = 1$.

The Fourier–Stieltjes transform of $\mu \in M(E^n)$ is given by

$$\mu^\vee(v) = \frac{1}{(2\pi)^{n/2}} \int_{E^n} e^{-i(v,x)}\, d\mu(x). \tag{1.2}$$

In saturation theory on $L^p(E^n)$, $1 < p \leqslant 2$, the classical Fourier transform method leads to the classes

$$V_\alpha{}^p = \{f(x) \in L^p(E^n);\ |v|^\alpha f^\wedge(v) = g^\wedge(v),\ g(x) \in L^p(E^n)\} \qquad (\alpha > 0) \tag{1.3}$$

which represent saturation classes of various approximation processes defined by singular integrals of convolution type (cf. Section 6) for specific values of α.

The characterization problem for the classes $V_\alpha{}^p$ in the case of several variables involves one essential difficulty (we choose $\alpha = 2$ for simplicity): If $f(x) \in V_2{}^p$ and if, in addition, the partial derivatives $\partial f/\partial x_j$, $\partial^2 f/\partial x_j{}^2$ $(j = 1, 2, \ldots, n)$ are assumed to exist and belong to L^p, then

$$\Delta f \equiv \sum_{j=1}^{n} \partial^2 f/\partial x_j{}^2 = -g(x).$$

But there seems to be no direct way to infer, under the lone assumption $f(x) \in V_2{}^p$, the existence of these derivatives or of functions $g_j(x) \in L^p$ with $v_j{}^2 f^\wedge(v) = g_j{}^\wedge(v)$ $(j = 1, 2, \ldots, n)$. One result of Section 3 states that $f(x) \in V_2{}^p$ if and only if $\Delta f = -g(x) \in L^p$, provided that Δf is understood in the distributional sense. Moreover, one obtains that even $\partial f/\partial x_j$, $\partial^2 f/\partial x_j \partial x_l$, $j, l = 1, 2, \ldots, n$ exist as elements of L^p.

In view of this result, the question arose whether there is a connection between $V_\alpha{}^p$ and the well-known Sobolev spaces for any integral α, and whether there is an analog for fractional α. Affirmative answers to these questions will be obtained, using the following procedure: At first we characterize $V_\alpha{}^p$ by classes of Bessel potentials in order, on the one hand, to be able to introduce distributional methods, and on the other hand, to obtain an extension of $V_\alpha{}^p$ for $p > 2$. These classes of Bessel potentials are, in turn, characterized by known theorems in terms of Sobolev spaces and several other equivalent conditions. These results then enable one to return from the distributional setting to classical conditions, e.g., to ordinary smoothness conditions.

There is another reason why the spaces of Bessel potentials are important here. An attempt to extend the class $V_\alpha{}^p$ from $1 < p \leqslant 2$ by mere interpretation of the Fourier transform in the distributional sense is easily seen to end in failure since even for arbitrarily smooth and rapidly decreasing functions $f(x)$ the product $|v|^\alpha f^\wedge(v)$ cannot be defined in the distributional sense in view of the fact that the functions $|v|^\alpha$ and $|v|^{-\alpha}$ do not belong to any one of the spaces S, S', $\mathcal{O}_M$, $\mathcal{O}_C{}'$. On the other hand, it is clear that the properties of the class $V_\alpha{}^p$ are determined only by the behavior of $|v|^\alpha$ for large $|v|$, and that any singularity at the origin is not significant for the definition of $V_\alpha{}^p$. Indeed, the class remains unchanged if we replace $|v|^\alpha$ by the function $(1 + |v|^2)^{\alpha/2}$ which has the same behavior at infinity as $|v|^\alpha$ but belongs to $\mathcal{O}_M$. Thus, the new class can also be defined for $p > 2$ by using the distributional Fourier transform, and we shall see in Section 6 that it is indeed the Favard class of the generalized singular integral of Weierstrass for $p > 2$ which is defined for $t, \kappa > 0; f(x) \in L^p(E^n)$, $1 \leqslant p < \infty$, by

$$W_t{}^\kappa(f; x) = (2\pi)^{-n/2} t^{-n/\kappa} \int_{E^n} f(x - u)\, H_\kappa(ut^{-1/\kappa})\, du, \tag{1.4}$$

where $H_\kappa(x)$ is given by $H_\kappa{}^\wedge(v) = e^{-|v|^\kappa}$. For $\kappa = 1$ or 2 this approximation process reduces to the singular integrals of Cauchy–Poisson and of Gauss–Weierstrass, respectively. The singular integral (1.4) will serve as a representative application of our results.

The actual aim of Section 6 is to present a distributional method in proving saturation theorems for n-dimensional singular integrals, in particular for the integral (1.4). The fundamental identity (6.2) will play an important role in both directions of the theorems. The method itself may be regarded as an immediate generalization of the functional or dual method employed in [*21*].

We conclude this section with a series of definitions and lemmas in the theory of distributions, the notation being that of Schwartz [*28*] and Friedman [*19*].

Let P be the set of all nonnegative integers and $P^n = \{k \in E^n; k_j \in P\}$. For $k \in P^n$ we define

$$D^k = (\partial/\partial x_1)^{k_1}(\partial/\partial x_2)^{k_2} \ldots (\partial/\partial x_n)^{k_n}, \tag{1.5}$$

and for such k we set $|k| = \sum_{j=1}^n k_j$, called the order of the operator D^k. In particular, we write

$$\nabla = \sum_{j=1}^{n} \sigma^j \frac{\partial}{\partial x_j}, \qquad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j{}^2}.$$

Here and in the sequel σ^j denotes the unit coordinate vector along the j-axis.

Furthermore, we define:

$C^\infty(E^n)$ is the set of infinitely differentiable functions on E^n, all of whose derivatives are bounded. D is that subset of functions in $C^\infty(E^n)$ which have compact support.

D' is the space of all continuous linear functionals (distributions) on D. The action of $f \in D'$ on $\varphi(x) \in D$ is written $\langle f, \varphi(x)\rangle$. The distributional derivative $D^k f$ of $f \in D'$ is defined by $\langle D^k f, \varphi(x)\rangle = (-1)^{|k|}\langle f, D^k\varphi(x)\rangle$ where $k \in P^n$ and $\varphi(x) \in D$.

$S = \{\varphi(x) \in C^\infty(E^n); \sup_{x \in E^n} |x|^s |D^k\varphi(x)| \leqslant C_{s,k}, s \in P, k \in P^n\}$ is the space of rapidly decreasing functions. A sequence $\{\varphi_m(x)\} \in S$ is said to converge in S to $\varphi(x) \in S$ if

$$\lim_{m \to \infty} |x|^s |D^k(\varphi_m(x) - \varphi(x))| = 0$$

uniformly in x for every $s \in P$, $k \in P^n$.

S' is the space of tempered distributions, i.e., of continuous linear functionals on S.

$\mathcal{O}_M$ is the space of infinitely differentiable functions on E^n which are slowly increasing, i.e., whose derivatives are each bounded by some power of $|x|$ as $|x| \to \infty$. $\mathcal{O}_C{}'$ is the space of rapidly decreasing distributions, i.e., of distributions T for which $(1 + |x|^2)^s T$, $s \in P$, is a "bounded distribution", thus a continuous linear functional on the space D_{L^1} of those $C^\infty(E^n)$-functions which, together with all their derivatives, belong to $L^1(E^n)$. $\mathcal{O}_M$ and $\mathcal{O}_C'$ are both subspaces of S'.

If $f(x)$ is a locally integrable function, then

$$\langle f, \varphi(x)\rangle = \int_{E^n} f(x)\,\varphi(x)\,dx \qquad (\varphi(x) \in S)$$

defines a distribution $f \in D'$. Distributions f of this type are called regular distributions.

(1.6) LEMMA. *If $f \in \mathcal{O}_C'$ and $g \in S'$, then the convolution $f * g$, defined by*

$$\langle f * g, \varphi(x)\rangle = (2\pi)^{-n/2}\langle f, \langle g, \varphi(x+u)\rangle\rangle \qquad (\varphi(x) \in S)$$

(this means that g is applied to $\varphi(x+u)$ with respect to x and f is applied to $\langle g, \varphi(x+u)\rangle$ with respect to u) exists as an element in S'.

The classical Fourier transform $\mathfrak{F}$ of (1.1) and its inverse

$$\mathfrak{F}^{-1}[\varphi](v) = (2\pi)^{-n/2}\int_{E^n} e^{i(v,x)}\,\varphi(x)\,dx$$

are continuous one-to-one mappings of S onto S, i.e., $\mathfrak{F}^{-1}\{\mathfrak{F}[\varphi(x)]\} = \varphi(x)$. If $f \in S'$, the distributional Fourier transform $f^\wedge$ is defined by

$$\langle f^\wedge, \varphi(x)\rangle = \langle f, \varphi^\wedge(v)\rangle \qquad (\varphi(x) \in S).$$

(1.7) LEMMA. *The distributional Fourier transform on S' and its inverse are continuous one-to-one mappings of S' onto S',* i.e., *for $f \in S'$ one has $\mathfrak{F}^{-1}\{\mathfrak{F}[f]\} = f$.*

(1.8) LEMMA. *The distributional Fourier transform and its inverse are continuous one-to-one mappings of $\mathcal{O}_M$ onto $\mathcal{O}_C'$.*

(1.9) LEMMA. *If $f \in S'$, then $\mathfrak{F}[\partial f/\partial x_j] = iv_j f^\wedge$.*

(1.10) LEMMA. *If $f(x) \in L^p(E^n)$, $1 \leqslant p \leqslant 2$, the ordinary and the distributional Fourier transform are equal.*

(1.11) LEMMA. *If $f, g \in S'$ and $f^\wedge = g^\wedge$, then $\langle f, \varphi(x)\rangle = \langle g, \varphi(x)\rangle$ for every $\varphi(x) \in S$. If f and g are, in addition, regular distributions, then $f(x) = g(x)$* a.e.

(1.12) LEMMA. *Let $f \in \mathcal{O}_C'$ and $g \in S'$. Then $f^\wedge \in \mathcal{O}_M$, $g^\wedge \in S'$, and*

$$(f * g) = f^\wedge g^\wedge \in S'.$$

(1.13) LEMMA. *Let $f \in S'$ and define the shifted distribution f_u by*

$$\langle f_u, \varphi(x)\rangle = \langle f, \varphi(x-u)\rangle.$$

Then

$$\lim_{t\to 0} t^{-\nu}\left\langle f_{t\sigma_j} - f - \sum_{\nu=1}^{m-1} \frac{t^\nu}{\nu!}\frac{\partial^\nu f}{\partial x_j{}^\nu}, \varphi(x)\right\rangle = \left\langle \frac{\partial^m f}{\partial x_j{}^m}, \varphi(x)\right\rangle$$

$$(m = 1, 2, \ldots;\ \varphi(x) \in S).$$

For proofs of these lemmas and details we refer to Schwartz [*28*]. Finally, we need the Taylor formula:

(1.14) LEMMA. *Let $f(x)$ be equal* a.e. *to a function $F(x)$ which, together with its partial derivatives up to the order $m-1$ $(m = 1, 2, \ldots)$, is locally absolutely continuous. Then one has* a.e.

$$f(x+h) - f(x) = \sum_{\nu=1}^{m-1} \frac{[(h,\nabla)^\nu F](x)}{\nu!} + \frac{1}{(m-1)!}\int_0^1 (1-\tau)^{m-1}[(h,\nabla)^m F]$$

$$\times (x + h\tau)\, d\tau.$$

2. BASIC THEOREMS

We begin with the definition and some properties of the Bessel kernel $G_\alpha(x)$ and of the space L_α^p of Bessel potentials.

(2.1) DEFINITION. Let $\alpha > 0$. The function

$$G_\alpha(x) = \{2^{(2-\alpha)/2}/\Gamma(\alpha/2)\}\,(|x|/2)^{-(n-\alpha)/2}\, K_{(n-\alpha)/2}(|x|)$$

is called the Bessel kernel, where

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(s) - I_\nu(s)}{\sin \pi\nu}, \qquad I_\nu(s) = \sum_{m=0}^{\infty} \frac{(s/2)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}$$

are the modified Bessel functions of order ν of the third and the first kind, respectively.

$G_\alpha(x)$ is nonnegative [*18*, p. 192], belongs to $L^1(E^n)$, and

$$\int_{E^n} G_\alpha(x)\, dx = (2\pi)^{n/2}, \tag{2.2}$$

its Fourier transform being given by $G_\alpha{}^\wedge(v) = (1 + |v|^2)^{-\alpha/2}$, [*2*, p. 417]. For other representations and further properties see [*1*, Chap. I, Section 1], [*2*, pp. 413–417], [*17*].

The convolution

$$(G_\alpha * h)(x) = (2\pi)^{-n/2} \int_{E^n} G_\alpha(x-u)\, h(u)\, du$$

with $h(x) \in L^p(E^n)$, $1 \leqslant p < \infty$, is called the Bessel potential of $h(x)$. It belongs to L^p and satisfies $\|(G_\alpha * h)(x)\|_p \leqslant \|h(x)\|_p$. In view of $G_\alpha^\wedge(v) \in \mathcal{O}_M$, it follows by Lemmas (1.8), (1.12) that

$$(G_\alpha * h)^\wedge = (1+|v|^2)^{-\alpha/2} h^\wedge, \tag{2.3}$$

where $h^\wedge$ is taken in the distributional sense.

(2.4) DEFINITION. Let $\alpha > 0$ and $1 \leqslant p < \infty$. The space

$$L_\alpha^p = \{f(x) \in L^p(E^n); f(x) = (G_\alpha * h)(x),\, h(x) \in L^p(E^n)\}$$

is called a space of Bessel potentials.

To obtain an equivalent definition, since $(1+|v|^2)^{\alpha/2} \in \mathcal{O}_M$, we may define a distribution $G_{-\alpha}$, $\alpha > 0$, by $G_{-\alpha}^\wedge = (1+|v|^2)^{\alpha/2}$. By Lemma (1.8) $G_{-\alpha} \in \mathcal{O}_C'$, and in view of Lemma (1.6), the distributional convolution $G_{-\alpha} * f$ exists for each $f \in S'$ as an element of S'. Therefore the following definition is meaningful:

$$L_\alpha^p = \{f \in S';\, G_{-\alpha} * f = h(x) \in L^p(E^n)\}. \tag{2.5}$$

This definition is equivalent to (2.4), as well as to

$$L_\alpha^p = \{f(x) \in L^p(E^n);\, (1+|v|^2)^{\alpha/2} f^\wedge = h^\wedge,\, h(x) \in L^p(E^n)\}. \tag{2.6}$$

Indeed, we note that Definition (2.4) immediately implies (2.6) which in turn is contained in (2.5) in view of the definition of $G_{-\alpha}$ and Lemma (1.11). If $G_{-\alpha} * f = h(x)$, $h(x) \in L^p$, then $(G_\alpha * (G_{-\alpha} * f))(x) = (G_\alpha * h)(x)$, and this is equal to $f(x)$ by Lemmas (1.12), (1.11). Thus $f(x) = (G_\alpha * h)(x)$, $h(x) \in L^p$, and this proves the equivalence of these three definitions.

As mentioned in the introduction, the definition of the class V_α^p cannot be extended for $p > 2$ by mere interpretation in the distributional sense. So we must first of all restrict the characterization of V_α^p to $1 < p \leqslant 2$. The following Lemma of Stein [30] is basic.

(2.7) LEMMA. *Let* $\alpha > 0$. *There exist measures* $\mu_\alpha^{(i)} \in M(E^n)$, $i = 1, 2, 3$ *such that*

$$(1+|v|^2)^{\alpha/2} = [\mu_\alpha^{(1)}]^\vee(v) + |v|^\alpha [\mu_\alpha^{(2)}]^\vee(v),$$

$$|v|^\alpha = (1+|v|^2)^{\alpha/2} [\mu_\alpha^{(3)}]^\vee(v).$$

This leads to the following fundamental

(2.8) THEOREM. *Let* $f(x) \in L^p(E^n)$, $1 < p \leqslant 2$, and $\alpha > 0$. *Then* $f(x) \in V_\alpha^p$ *if and only if* $f(x) \in L_\alpha^p$.

Proof. If $f(x) \in V_\alpha{}^p$, by Lemma (1.6) we define the distribution $h = G_{-\alpha} * f \in S'$, whose Fourier transform is $h^\wedge = (1 + |v|^2)^{\alpha/2} f^\wedge(v)$, by Lemma (1.12). Now Lemma (2.7) and the hypothesis yield

$$h^\wedge = [\mu_\alpha^{(1)}]^\vee(v) f^\wedge(v) + [\mu_\alpha^{(2)}]^\vee(v) g^\wedge(v) = [f * d\mu_\alpha^{(1)}]^\wedge(v) + [g * d\mu_\alpha^{(2)}]^\wedge(v)$$

by the convolution theorem of the ordinary Fourier transform. By Lemma (1.11) we see that h is regular,

$$h(x) = (f * d\mu_\alpha^{(1)})(x) + (g * d\mu_\alpha^{(2)})(x) \quad \text{a.e.},$$

and

$$\|h\|_p \leqslant \|f\|_p \, \|\mu_\alpha^{(1)}\|_M + \|g\|_p \, \|\mu_\alpha^{(2)}\|_M.$$

Thus $h(x) \in L^p$ or $f(x) \in L_\alpha{}^p$ according to (2.5).

Conversely, let $f(x) \in L_\alpha{}^p$. By (2.6) and Lemma (2.7), with $h(x) \in L^p$,

$$|v|^\alpha f^\wedge(v) = [\mu_\alpha^{(3)}]^\vee(v)(1 + |v|^2)^{\alpha/2} f^\wedge(v) = [\mu_\alpha^{(3)}]^\vee(v) h^\wedge(v).$$

As above one obtains $|v|^\alpha f^\wedge(v) = g^\wedge(v)$ with $g(x) = (h * d\mu_\alpha^{(3)})(x) \in L^p$, and thus $f(x) \in V_\alpha{}^p$.

We mention that in case $p = 2$ the proof is rather simple in view of Plancherel's theorem and the fact that the class of multipliers (L^2, L^2) consists of all bounded measurable functions.

With the aid of Theorem (2.8) it is now easy to make use of known results on Sobolev spaces in order to obtain characterizations of $V_\alpha{}^p$ for integral α.

(2.9) Definition. Let $1 < p < \infty$ and $\alpha = 1, 2, \ldots$. The space

$$W_\alpha{}^p = \{f(x) \in L^p(E^n);\ D^k f \in L^p(E^n) \qquad \text{for every } |k| \leqslant \alpha\},$$

$D^k f$ being the distributional derivative, is called Sobolev space of order α.

The following lemma has been shown by Calderón [*17*, p. 36] and Aronszajn, Mulla and Szeptycki [*1*, Theorem 11.1].

(2.10) Lemma. *Let* $\alpha = 1, 2, \ldots$ *and* $1 < p < \infty$. *Then* $f(x) \in L_\alpha{}^p$ *if and only if* $f(x) \in W_\alpha{}^p$.

Thus for integral α one has $f(x) \in V_\alpha{}^p$ if and only if $f(x) \in W_\alpha{}^p$.

It is the purpose of the next two sections to treat the cases when α is even or odd, separately and in all details.

3. The Case α Even

This case is the simpler one, for in case $\alpha = 2m$, $m \in P$, the class $V_\alpha{}^p$ can be extended for $p > 2$ by just replacing the ordinary Fourier transform by the distributional one (since $|v|^{2m} \in \mathcal{O}_M$). Moreover, the classes defined by

$|v|^{2m}f^\wedge = g^\wedge$ and $(iv)^{2m}f^\wedge = g^\wedge$, respectively, are identical, which suggests that characterizations of V_{2m}^p are possible by ordinary Lipschitz conditions.

We introduce the following notation:

(3.1) DEFINITION. $AC^{r-1}(L^p)$, $r = 1, 2, \ldots$, denotes the space of those functions $f(x) \in L^p(E^n)$ for which there exists a function $F(x)$ such that $f(x) = F(x)$ a.e. and $F(x)$ together with all of its ordinary partial derivatives $(D^k F)(x)$ of order $|k| \leqslant r-1$ are locally absolutely continuous in each variable, and

$$(D^k F)(x) \in L^p(E^n) \qquad \text{for every } |k| \leqslant r.$$

We first treat the case $\alpha = 2$.

(3.2) THEOREM. *Let $f(x) \in L^p(E^n)$, $1 < p < \infty$. The following assertions are equivalent:*

(a) $f(x) \in V_2{}^p \equiv \{f(x) \in L^p(E^n); |v|^2 f^\wedge = g^\wedge, g(x) \in L^p(E^n)\}$ *where, for $p > 2$, the Fourier transform is taken in the distributional sense;*

(b) $f(x) \in L_2{}^p \equiv \{f(x) \in L^p(E^n); (1+|v|^2)f^\wedge = h^\wedge, h(x) \in L^p(E^n)\}$;

(c) $f(x) \in W_2{}^p$;

(d) $\Delta f = g(x)$ *for a function $g(x) \in L^p(E^n)$, where the operator Δ is taken in the distributional sense;*

(e) $f(x) \in AC^1(L^p)$;

(f) *the derivatives $\partial f/\partial x_j$, $\partial^2 f/\partial x_l x_j$ exists as limits in norm*, i.e.,

$$\lim_{t\to 0}\left\| \frac{f(x+t\sigma^j)-f(x)}{t} - \frac{\partial f}{\partial x_j}(x)\right\|_p = 0$$

$$\lim_{t\to 0}\left\| \frac{\dfrac{\partial f}{\partial x_j}(x+t\sigma^l) - \dfrac{\partial f}{\partial x_j}(x)}{t} - \frac{\partial^2 f}{\partial x_l\,\partial x_j}(x)\right\|_p = 0 \qquad (j, l = 1, 2, \ldots, n);$$

(g) $f(x) \in AC^0(L^p)$, *and*

$$\left\| f(x+t\sigma^j) - f(x) - t\frac{\partial F}{\partial x_j}(x)\right\|_p = O(t^2) \qquad (t \to 0;\, j = 1, 2, \ldots, n);$$

(h) $f(x) \in \mathrm{Lip}^*(2, p; \sigma^j)$ *for $j = 1, 2, \ldots, n$, that is*

$$\|f(x+2t\sigma^j) - 2f(x+t\sigma^j) + f(x)\|_p = O(t^2) \qquad (t \to 0);$$

(i) $f(x) \in \mathrm{Lip}\,(2, p)$, i.e.,

$$\|f(x+2h) - 2f(x+h) + f(x)\|_p = O(|h|^2) \qquad (|h| \to 0).$$

Proof. We proceed as follows. First of all we establish the assertions (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a), then (c) $\Rightarrow$ (e) $\Rightarrow$ (g) $\Rightarrow$ (d) and (e) $\Rightarrow$ (f) $\Rightarrow$ (c), and finally (e) $\Rightarrow$ (i) $\Rightarrow$ (h) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b): trivial. (b) $\Rightarrow$ (c): by Lemma (2.10). (c) $\Rightarrow$ (d): trivial. (d) $\Rightarrow$ (a): by Lemma (1.9). (c) $\Rightarrow$ (e): $f(x) \in W_2^p$ means that the distributional derivatives of $f(x)$ up to the second order exist as elements of L^p, i.e., $\partial f/\partial x_j = g_j(x)$ and $\partial^2 f/\partial x_l \partial x_j = h_{l,j}(x) \in L^p$ for $j, l = 1, 2, \ldots, n$. Now, by Lemma (1.9), $iv_j f^\wedge = g_j^\wedge$ and $(iv_l)(iv_j) f^\wedge = h_{l,j}^\wedge$. The standard proof that $iv_j f^\wedge(v) = g_j^\wedge(v)$ with $g_j(x) \in L^p$, $1 < p \leqslant 2$, implies that $f(x)$ is equal a.e. to a function $F(x)$ which is locally absolutely continuous in x_j (cf. [*25*, p. 126]) can also be employed for $p > 2$: Indeed, if $iv_j f^\wedge = g_j^\wedge$, then for each $\varphi(x) \in S$

$$\langle [f(x + t\sigma^j) - f(x)]^\wedge, \varphi(v) \rangle = \left\langle \frac{e^{iv_j t} - 1}{iv_j} g_j^\wedge, \varphi(v) \right\rangle. \tag{3.3}$$

On the other hand, as $g_j(x) \in L^p$,

$$\int_0^t g_j(x + s\sigma^j)\, ds$$

exists as a function in L^p, and thus

$$\left\langle \int_0^t g_j(x + s\sigma^j)\, ds, \varphi^\wedge(x) \right\rangle = \int_{E^n} \left\{ \int_0^t g_j(x + s\sigma^j)\, ds \right\} \varphi^\wedge(x)\, dx.$$

The Theorem of Fubini can be applied, giving

$$\begin{aligned} \left\langle \left[\int_0^t g_j(x + s\sigma^j)\, ds \right]^\wedge, \varphi(v) \right\rangle &= \int_0^t \langle g_j(x + s\sigma^j), \varphi^\wedge(x) \rangle\, ds \\ &= \int_0^t \langle e^{iv_j s} g_j^\wedge, \varphi(v) \rangle\, ds. \end{aligned}$$

In order to evaluate the latter integral we use a theorem on the integration of distributions depending continuously upon a parameter [*34*, p. 76] and obtain

$$\left\langle \left[\int_0^t g_j(x + s\sigma^j)\, ds \right]^\wedge, \varphi(v) \right\rangle = \left\langle \frac{e^{iv_j t} - 1}{iv_j} g_j^\wedge, \varphi(v) \right\rangle.$$

In view of (3.3) and Lemma (1.11) this yields for almost every x and t:

$$f(x + t\sigma^j) - f(x) = \int_0^t g_j(x + s\sigma^j)\, ds. \tag{3.4}$$

Now it follows by a routine argument (cf. [*21*, Section 5]) that there exists an $F(x) = f(x)$ a.e. for which this equality holds for every x and t, thus $F(x)$ is absolutely continuous in x_j, $j = 1, 2, \ldots, n$.

For the proof that, furthermore, all partial derivatives of the first order are absolutely continuous in each variable and that all partial derivatives of the second order belong to L^p we only have to apply the same argument to the remaining relation $(iv_l) g_j^\wedge = h_{l,j}^\wedge$, and (e) follows.

(e) ⇒ (g): By Taylor's formula we have

$$f(x+t\sigma^j)-f(x)-t\frac{\partial F}{\partial x_j}(x)=t^2\int_0^1(1-\tau)\frac{\partial^2 F}{\partial x_j^2}(x+t\tau\sigma^j)\,d\tau,$$

and this implies (g) by the generalized Minkowski inequality.

(g) ⇒ (d): By Lemma (1.13) we have for the distributional derivative $\partial^2 f/\partial x_j^2$ that for each $\varphi(x)\in S$

$$\lim_{t\to 0} t^{-2}\left\langle f(x+t\sigma^j)-f(x)-t\frac{\partial F}{\partial x_j}(x),\varphi(x)\right\rangle=\left\langle\frac{\partial^2 f}{\partial x_j^2},\varphi(x)\right\rangle.$$

On the other hand, the assumption (g) and the weak compactness of the space L^p imply the existence of a function $g_j(x)\in L^p$ and a subsequence $\{t_r\}$ such that for each $\varphi(x)\in S$

$$\lim_{r\to\infty}\left\langle f(x+t_r\sigma^j)-f(x)-t_r\frac{\partial F}{\partial x_j}(x),\varphi(x)\right\rangle=\langle g_j(x),\varphi(x)\rangle,$$

and this yields (d). (Instead of (g) ⇒ (d) one might as well have concluded (g) ⇒ (a).)

(e) ⇒ (f): We have only to prove the second limit-relation since the first one is a trivial consequence of (g). In the step (c) ⇒ (e) we first deduced (3.4) from $(iv_j)f^\wedge=g_j^\wedge$ which means that $\partial F/\partial x_j=g_j(x)$ with $F(x)=f(x)$ a.e., and then applied the same argument to the relation $(iv_l)g_j^\wedge=h_{l,j}^\wedge$. Thus we also obtain

$$\frac{\partial F}{\partial x_j}(x+t\sigma^l)-\frac{\partial F}{\partial x_j}(x)=\int_0^t h_{l,j}(x+s\sigma^l)\,ds$$

which implies (f).

The remaining steps of the proof proceed along similar lines. (f) ⇒ (c) follows by Lemma (1.13), and (e) ⇒ (i) is a consequence of the formula

$$F(x+2h)-2F(x+h)+F(x)=\int_{-1}^1(1-|\tau|)\,[(h,\nabla)^2F](x+h+h\tau)\,d\tau.$$

(i) ⇒ (h) is trivial, and (h) ⇒ (c) follows as in (g) ⇒ (d) by the weak compactness of L^p.

Remark. The proof shows that also the following statement is equivalent to (a):

(a′) There exist functions $g_j(x)$, $h_{l,j}(x)\in L^p(E^n)$ such that

$$(iv_j)f^\wedge=g_j^\wedge,\ (iv_l)(iv_j)f^\wedge=h_{l,j}^\wedge\qquad(j,l=1,2,\dots,n).$$

Instead of (e) we might also have used the following weaker statement:

(e′) $f(x)$ is equal a.e. to a function $F(x)$ which, together with its ordinary derivatives $\partial F/\partial x_j$, is locally absolutely continuous in x_j, and

$$\partial F/\partial x_j,\ \partial^2 F/\partial x_j{}^2 \in L^p(E^n) \qquad (j = 1, 2, \ldots, n).$$

Note that in assertion (e′) nothing is assumed about the mixed derivatives.

Theorem (3.2) gives thus a positive answer to the question raised in Section 1, whether for instance the hypothesis $f(x) \in V_2{}^p$ implies the existence of all ordinary partial derivatives of $f(x)$ up to the second order as elements of L^p.

In the general case $\alpha = 2m$, $m = 2, 3, \ldots$ we have the following theorem:

(3.5) THEOREM. *Let* $f(x) \in L^p(E^n)$, $1 < p < \infty$, and $\alpha = 2m$, $m = 1, 2, \ldots$. *The following assertions are equivalent:*

(a) $f(x) \in V_{2m}^p \equiv \{f(x) \in L^p(E^n);\ |v|^{2m} f^\wedge = g^\wedge,\ g(x) \in L^p(E^n)\}$ *where, for* $p > 2$, *the Fourier transform is taken in the distributional sense;*

(b) $f(x) \in L_{2m}^p$;

(c) $f(x) \in W_{2m}^p$;

(d) $\Delta^m f = g(x)$ *for a function* $g(x) \in L^p(E^n)$, *the operator* $\Delta^m = \Delta(\Delta^{m-1})$ *being taken in the distributional sense;*

(e) $f(x) \in AC^{2m-1}(L^p)$;

(f) *all derivatives of* $f(x)$ *up to an order* $2m$ *exist as limits in norm;*

(g) $f(x) \in AC^{2m-2}(L^p)$, *and for* $t \to 0$, $j = 1, 2, \ldots, n$

$$\left\| f(x + t\sigma^j) - f(x) - \sum_{\nu=1}^{2m-1} \frac{t^\nu}{\nu!} \frac{\partial^\nu F}{\partial x_j{}^\nu}(x) \right\|_p = O(t^{2m});$$

(h) $\|\Delta_{t\,\sigma(1)}^2 \Delta_{t\,\sigma(2)}^2 \cdots \Delta_{t\,\sigma(m)}^2 f(x)\|_p = O(t^{2m}) \qquad (t \to 0),$

for every choice of unit coordinate vectors $\sigma_{(1)}, \sigma_{(2)}, \ldots, \sigma_{(m)}$.

Proof. Most of the arguments are the same as in the proof of Theorem (3.2), so we consider only the steps which require some further explanation.

(a) ⇒ (b): the proof is the same as in Theorem (2.8).

(e) ⇒ (h): follows by repeated application of Lemma (1.14).

(h) ⇒ (a): as in step (g) ⇒ (d) of Theorem (3.2) the hypothesis and the weak compactness of the space L^p yield the existence of a sequence $\{t_\nu\}$ with $\lim_{\nu\to\infty} t_\nu = 0$ and a function $g(x) \in L^p$ depending on the special choice of $\sigma_{(j)}$, $j = 1, 2, \ldots, m$ such that for every $\varphi(x) \in S$

$$\lim_{\nu\to\infty} t_\nu^{-2m} \langle \Delta_{t_\nu\sigma(1)}^2 \Delta_{t_\nu\sigma(2)}^2 \cdots \Delta_{t_\nu\sigma(3)}^2 f(x), \varphi(x)\rangle = \langle g(x), \varphi(x)\rangle.$$

Replacing $\varphi(x)$ by $\varphi^\wedge(x)$ we obtain in view of the definition of the distributional Fourier transform

$$\lim_{\nu\to\infty}\left\langle\left\{\frac{e^{it_\nu(v,\sigma_{(1)})}-1}{t_\nu}\right\}^2\cdots\left\{\frac{e^{it_\nu(v,\sigma_{(m)})}-1}{t_\nu}\right\}^2 f^\wedge,\varphi(v)\right\rangle=\langle g^\wedge,\varphi(v)\rangle.$$

Now $t_\nu^{-1}\{e^{it_\nu(v,\sigma_{(j)})}-1\}$ converges to $i(v,\sigma_{(j)})$ in $\mathcal{O}_M$, thus the limit is

$$\langle[i(v,\sigma_{(1)})\,i(v,\sigma_{(2)})\dots i(v,\sigma_{(m)})]^2 f^\wedge,\varphi(v)\rangle=\langle g^\wedge,\varphi(v)\rangle\qquad(\varphi(x)\in S).$$

As this holds for every choice of the vectors $\sigma_{(j)}$, one obtains corresponding relations for all products of even powers of $v_j, j=1,2,\dots,n$, the sum of whose exponents is $2m$. Thus each term of the binomial expansion of

$$|v|^{2m}f^\wedge=(\sum_{j=1}^n v_j^2)^m f^\wedge$$

is the distributional Fourier transform of some function in L^p, and (a) follows.

Finally we add another characterization in case $\alpha=2$ which we owe to a suggestion of Berens (see also [*11*, p. 278]).

(3.6) THEOREM. *Let* $f(x)\in L^p(E^n)$, $1<p<\infty$. *Then the assertions of Theorem* (3.2) *are equivalent to*

$$\left\|\sum_{k,k_j=\pm1}[f(x+tk)-f(x)]\right\|_p=O(t^2)\qquad(t\to0).$$

Proof. The term $\sum_{k,k_j=\pm1}f(x+tk)$ represents the sum of values $f(x+y)$ where y runs over all 2^n vertices of an n-dimensional cube with edge-length $2h$ centered at the origin. For brevity we write

$$\sum_{k,k_j=\pm1}[f(x+tk)-f(x)]\equiv\Box_t f(x).$$

Let condition (e) of Theorem (3.2) be satisfied. Then by Taylor's formula

$$f(x+tk)-f(x)=t[(k,\nabla)F](x)+t^2\int_0^1(1-\tau)[(k,\nabla)^2F](x+t\tau k)\,d\tau.$$

Summing now over $k,k_j=\pm1$, and combining those terms whose k-values differ only by sign, we obtain

$$\|\Box_t f(x)\|_p=t^2\left\|\sum_{k,k_j=\pm1}\int_0^1(1-\tau)[(k,\nabla)^2F](x+t\tau k)\,d\tau\right\|_p$$

$$\leqslant t^2 2^n\sum_{k,l=1}^n\left\|\frac{\partial^2F}{\partial x_l\,\partial x_j}\right\|_p=O(t^2)\qquad(t\to0).$$

Conversely, if $\|\square_t f(x)\|_p = O(t^2)$, we have on the one hand (cf. Lemma (1.13))

$$\lim_{t\to 0} \langle t^{-2} \square_t f(x), \varphi(x)\rangle = \langle \Delta f, \varphi(x)\rangle \qquad (\varphi(x) \in S),$$

and, on the other, by the weak compactness of L^p, that

$$\lim_{r\to\infty} \langle t_r^{-2} \square_{t_r} f(x), \varphi(x)\rangle = \langle g(x), \varphi(x)\rangle \qquad (\varphi(x) \in S)$$

for a subsequence $\{t_r\}$ and some $g(x) \in L^p$; thus condition (d) of Theorem (3.2) is satisfied, and the proof is complete.

4. The Case α Odd

The characterizations of $V_\alpha{}^p$ in the case α odd are not as simple as for α even since the function $|v|^{2m-1}$, $m = 1, 2, \ldots$, does not belong to $\mathcal{O}_M$. Here we obtain equivalent conditions upon the Riesz transforms (Hilbert transforms) of $f(x) \in L^p(E^n)$ defined by

$$f_j{}^\sim(x) \equiv [H_j f](x) = \lim_{\epsilon\to 0+} f_{j,\epsilon}^\sim(x),$$
$$f_{j,\epsilon}^\sim(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{|x-u|\geq\epsilon} f(u) \frac{x_j - u_j}{|x-u|^{n+1}}\, du \qquad (j = 1, 2, \ldots, n). \tag{4.1}$$

In the sequel, H denotes the vector with coordinates $f_j{}^\sim$:

$$[Hf](x) = \sum_{j=1}^{n} f_j(x)\, \sigma^j. \tag{4.2}$$

If $f(x) \in L^p(E^n)$, $1 < p < \infty$, the Riesz transforms $f_j{}^\sim(x)$ exist a.e. and belong to $L^p(E^n)$. Furthermore, $f_j{}^\sim(x)$ can be represented as limit in L^p-norm of the "conjugate Poisson integral" of $f(x)$:

$$\lim_{t\to 0+} \|[W_t{}^1(f;u)]_j{}^\sim(x) - f_j{}^\sim(x)\|_p = 0, \tag{4.3}$$

where

$$[W_t{}^1(f;u)]_j{}^\sim(x) = (f * q_{j,t})(x); \; q_{j,t}(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{(t^2 + |x|^2)^{(n+1)/2}} \; (t > 0).$$

We mention that the conjugate Poisson kernel $q_{j,t}(x)$ belongs to $L^q(E^n)$ for $q > 1$.

Another representation of $f_j{}^\sim(x)$ for $f(x) \in L^p$ is given by

$$\lim_{\epsilon\to 0+} \|f_j{}^\sim(x) - f_{j,\epsilon}^\sim(x)\|_p = 0.$$

Further, we need the following Parseval formulas for $f(x) \in L^p(E^n)$, $g(x) \in L^{p'}(E^n)$:

$$\text{(i)}\ \int_{E^n} f_j^{\sim}(x) g(x)\, dx = -\int_{E^n} f(x) g_j^{\sim}(x)\, dx,$$

$$\text{(ii)}\ (f_j^{\sim} * g)(x) = (f * g_j^{\sim})(x), \tag{4.4}$$

$$\text{(iii)}\ \int_{E^n} f(x)\overline{g(x)}\, dx = \sum_{j=1}^{n} \int_{E^n} f_j^{\sim}\ x)\overline{g_j^{\sim}(x)}\, dx,$$

$$\text{(iv)}\ (H, H)f(x) \equiv \sum_{j=1}^{n} [H_j[H_j f]](x) = -f(x) \text{ a.e.}$$

If $1 < p \leqslant 2$, the ordinary Fourier transform of $f_j^{\sim}(x)$ has the property

$$[f_j^{\sim}](v)^{\wedge} = -\frac{iv_j}{|v|} f^{\sim}(v) \text{ a.e.} \tag{4.5}$$

For proofs of these statements we refer to [*14*], cf. also [*33*] and [*24*].

(4.6) Lemma. *Let $f(x) \in L^p(E^n)$, $1 < p < \infty$. Then $f(x) \in L_1^p$ if and only if $f_j^{\sim}(x) \in L_1^p$ for $1 \leqslant j \leqslant n$.*

Proof. If $f(x) \in L_1^p$, then by (2.4) there is an $h(x) \in L^p$ such that

$$f(x) = (G_1 * h)(x).$$

Thus by Fubini's theorem

$$[W_t^1(f; u)]_j^{\sim}(x) = ((G_1 * h) * q_{j,t})(x) = (G_1 * (h * q_{j,t}))(x)$$
$$= (G_1 * [W_t^1(h; u)]_j^{\sim})(x).$$

For $t \to 0+$ this furnishes $f_j^{\sim}(x) = (G_1 * h_j^{\sim})(x)$ with $h_j^{\sim}(x) \in L^p$ in view of (4.3). Thus $f_j^{\sim}(x) \in L_1^p$ for $j = 1, 2, \ldots, n$.

Conversely, by (4.4), (iv), it is sufficient to apply the same argument to $f_j^{\sim}(x)$ in order to deduce $f(x) \in L_1^p$.

Now we have the following characterizations of L_1^p.

(4.7) Theorem. *Let $f(x) \in L^p(E^n)$, $1 < p < \infty$. The following statements are equivalent:*

(a) $f(x) \in L_1^p$;

(b) $f(x) \in W_1^p$;

(c) $(\operatorname{div} Hf) \equiv [\sum_{j=1}^{n} (\partial f_j^{\sim}/\partial x_j)] \in L^p(E^n)$, *where* $\partial/\partial x_j$ *means the distributional derivative;*

(d) $f_j^{\sim}(x) \in AC^0(L^p) \qquad (j = 1, 2, \ldots, n)$;

(e) $\|f_j^{\sim}(x + t\sigma^l) - f_j^{\sim}(x)\|_p = O(|t|) \qquad (t \to 0; j, l = 1, 2, \ldots, n)$;

(f) $\|f_j^{\sim}(x + h) - f_j^{\sim}(x)\|_p = O(|h|) \qquad (|h| \to 0; j = 1, 2, \ldots, n)$.

If $1 < p \leqslant 2$, *these statements are equivalent to:*

(g) *there exist functions* $g_{l,j}(x) \in L^p(E^n)$, $j, l = 1, 2, \ldots, n$, *such that*

$$(v_l v_j/|v|) f^{\wedge}(v) = g_{l,j}^{\wedge}(v);$$

(h) $f(x) \in V_1{}^p$.

Proof. The assertions (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (h) are valid in view of Lemma (2.10) and Theorem (2.8), respectively. It remains to show (b) $\Rightarrow$ (c) $\Rightarrow$ (a), (b) $\Rightarrow$ (d) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (b), and (d) $\Rightarrow$ (g) $\Rightarrow$ (h).

(b) $\Rightarrow$ (c): obvious in view of Lemma (4.6).

(c) $\Rightarrow$ (a): We begin with the case $1 < p \leqslant 2$. The hypothesis (c) states that the individual terms of the sum $\sum_{j=1}^{n} \partial f_j^{\sim}/\partial x_j$ are distributions and that only the whole sum is a regular distribution generated by a function in L^p. By (4.5) we have $[f_j^{\sim}]^{\wedge}(v) = -(iv_j/|v|) f^{\wedge}(v)$ in the classical sense. On the other hand, if we consider $f_j^{\sim}(x) \in S'$, we obtain $\partial f_j^{\sim}/\partial x_j \in S'$ and $[\partial f_j^{\sim}/\partial x_j] = iv_j [f_j^{\sim}]^{\wedge}(v)$. Hence

$$[\operatorname{div} Hf]^{\wedge}(v) = \sum_{j=1}^{n} [\partial f_j^{\sim}/\partial x_j]^{\wedge} = |v| f^{\wedge}(v) \tag{4.8}$$

or $f(x) \in V_1{}^p$, and this implies (a) by Theorem (2.8).

Now let $2 < p < \infty$. We have to show that $f(x) = (G_1 * h)(x)$ for some $h(x) \in L^p$. We assert that

$$h(x) = (f * d\mu_\alpha^{(1)})(x) + (g * d\mu_\alpha^{(2)})(x),$$

where $g(x) = (\operatorname{div} Hf)(x)$ and $\mu_\alpha^{(i)}$, $i = 1, 2$, are the bounded measures of Lemma (2.7). This function $h(x)$ is in L^p, and for $\varphi(x) \in S$, $\mu_\alpha^{(i)*}(x) = \mu_\alpha^{(i)}(-x)$, we have

$$\begin{aligned} \langle (G_1 * h)(x), \varphi(x) \rangle &= \langle (G_1 * f * d\mu_\alpha^{(1)})(x), \varphi(x) \rangle \\ &\quad + \langle (G_1 * g * d\mu_\alpha^{(2)})(x), \varphi(x) \rangle \\ &= \langle f(x), (\varphi * G_1 * d\mu_\alpha^{(1)*})(x) \rangle \\ &\quad + \langle g(x), (\varphi * G_1 * d\mu_\alpha^{(2)*})(x) \rangle \end{aligned} \tag{4.9}$$

by Fubini's theorem. Setting $\psi(x) = (\varphi * G_1 * d\mu_\alpha^{(2)*})(x)$, then also $\psi(x) \in S$, and

$$[\partial\psi/\partial x_j]_j^{\sim}(x) = (\partial\psi_j^{\sim}/\partial x_j)(x)$$

since the Fourier transforms of both sides are equal. Thus, by (4.4),

$$\langle g(x), \psi(x)\rangle = \sum_{j=1}^{n} \langle \partial f_j^{\sim}/\partial x_j, \psi(x)\rangle = -\sum_{j=1}^{n} \langle f_j^{\sim}(x), \partial\psi/\partial x_j\rangle$$

$$= \sum_{j=1}^{n} \int_{E^n} f(x)\,[\partial\psi/\partial x_j]_j^{\sim}(x)\,dx = \sum_{j=1}^{n} \int_{E^n} f(x)\,(\partial\psi_j^{\sim}/\partial x_j)(x)\,dx$$

$$= \int_{E^n} f(x) \left\{ \sum_{j=1}^{n} \partial/\partial x_j[\varphi_j^{\sim} * G_1 * d\mu_\alpha^{(2)*}](x) \right\} dx.$$

The last step followed by the boundedness of H_j and the convolution operation on L^p, $1 < p < \infty$. We then obtain by (4.9)

$$\begin{aligned} \langle (G_1 * h)(x), \varphi(x)\rangle &= \langle f(x), (\varphi * G_1 * d\mu_\alpha^{(1)*})(x) \\ &\quad + ([\operatorname{div} H\varphi] * G_1 * d\mu_\alpha^{(2)*})(x)\rangle \\ &= \langle f(x), (G_1 * H_\varphi)(x)\rangle \qquad (\varphi(x) \in S) \end{aligned} \tag{4.10}$$

with $H_\varphi(x) = (\varphi * d\mu_\alpha^{(1)*})(x) + ([\operatorname{div} H\varphi] * d\mu_\alpha^{(2)*})(x)$.
On the other hand, we have $\varphi(x) \in S \subset L_1{}^q$ for $1 < q \leqslant 2$ since

$$(1 + |v|^2)^{1/2} \varphi^\wedge(v) \in S$$

(cf. (2.6)). Thus equation (4.8) and Lemma (2.7) yield

$$\begin{aligned} H_\varphi{}^\wedge(v) &= \varphi^\wedge(v)\,[\mu_\alpha^{(1)}]^\vee(-v) + |v|\,\varphi^\wedge(v)\,[\mu_\alpha^{(2)}]^\vee(-v) \\ &= (1 + |v|^2)^{1/2} \varphi^\wedge(v) = [G_{-1} * \varphi]^\wedge(v) \end{aligned}$$

and $(G_1 * H_\varphi)(x) = \varphi(x)$ by the definition of G_{-1}. Therefore

$$\langle (G_1 * h)(x), \varphi(x)\rangle = \langle f(x), \varphi(x)\rangle$$

and so $f(x) \in L_1{}^p$.

(b) ⇒ (d): By (a) and Lemma (4.6) we have $f_j^{\sim}(x) \in W_1{}^p$. Therefore the distributional derivatives $\partial f_j^{\sim}/\partial x_l = g_{l,j}(x)$ belong to L^p. Hence the first step of the proof of Theorem (3.2), (c) ⇒ (e), applies, $f(x)$ being replaced by $f_j^{\sim}(x)$. In particular, we obtain by (3.4), for every x and t

$$F(x + t\sigma^l) - F(x) = t \int_0^1 g_{l,j}(x + st\sigma^l)\,ds \tag{4.11}$$

where $F(x) = f_j^{\sim}(x)$ a.e.

(d) $\Rightarrow$ (f): This is an immediate consequence of (4.11) since

$$\||h|^{-1}\{f_j^\sim(x+h)-f_j^\sim(x)\}\|_p = \left\|\sum_{l=1}^{n} |h|^{-1}\left\{f_j^\sim\left(x+\sum_{\nu=l+1}^{n} h_\nu\sigma^\nu + h_l\sigma^l\right)\right.\right.$$

$$\left.\left. -f_j^\sim\left(x+\sum_{\nu=l+1}^{n} h_\nu\sigma^\nu\right)\right\}\right\|_p$$

$$\leqslant \sum_{l=1}^{n}\frac{|h_l|}{|h|}\int_0^1\left\|g_{l,j}\left(x+\sum_{\nu=l+1}^{n} h_\nu\sigma^\nu + sh_l\sigma^l\right)\right\|_p ds$$

$$= O(1) \qquad (|h|\to 0).$$

(f) $\Rightarrow$ (e): trivial.

(e) $\Rightarrow$ (b): This is proved as in step (g) $\Rightarrow$ (c) of the proof of Theorem (3.2). Finally, the steps (d) $\Rightarrow$ (g) $\Rightarrow$ (h) are trivial consequences of (4.5). This proves Theorem (4.7) completely.

Remark. The following statements are also equivalent to (a):

(e′) $\|f(x+t\sigma^l)-f(x)\|_p = O(|t|) \qquad (t\to 0;\, l=1,2,\ldots,n)$;

(f′) $\|f(x+h)-f(x)\|_p = O(|h|) \qquad (|h|\to 0;\, l=1,2,\ldots,n)$;

(i) the derivatives $\partial f_j^\sim/\partial x_l$ exist as limits in norm, i.e.,

$$\lim_{t\to 0}\|t^{-1}\{f_j^\sim(x+t\sigma^l)-f_j^\sim(x)\}-(\partial f_j^\sim/\partial x_l)(x)\|_p = 0 \qquad (j,l=1,2,\ldots,n).$$

The proof of (e) $\Rightarrow$ (e′) and (f) $\Rightarrow$ (f′) is obvious in view of the boundedness of the operator H_j on L^p for $1<p<\infty$ and formula (4.4) which imply e.g.,

$$\|f(x+t\sigma^l)-f(x)\|_p = \left\|\sum_{j=1}^{n}\{[H_j(H_jf)](x+t\sigma^l)-[H_j(H_jf)](x)\}\right\|_p$$

$$\leqslant A_p\sum_{j=1}^{n}\|f_j^\sim(x+t\sigma^l)-f_j^\sim(x)\|_p = O(|t|) \qquad (t\to 0),$$

where A_p depends only on p.

The statement (d) $\Rightarrow$ (i) can be shown as in step (e) $\Rightarrow$ (g) of the proof of Theorem (3.2), and (i) $\Rightarrow$ (b) is trivial.

Theorem (4.7) may also be stated for the general case α odd and the corresponding proofs can be carried over with suitable modifications.

5. The General Case $\alpha > 0$

If α is fractional, the question arises whether e.g. an analog of Lemma (2.10) remains true for suitable "fractional" Sobolev spaces. In general, the answer is positive only for $p=2$, and here we can use known results of Stein, Nikolskiĭ, Aronszajn, Mulla and Szeptycki, Besov, and Taibleson. In the following definitions we confine ourselves to the case $p = 2$. Let $\alpha > 0$ and $\alpha = m + \beta$ with m integral and $0 \leqslant \beta < 1$. For $0 < \beta < 1$, the Sobolev space $W_\alpha{}^2$ (defined for $\beta = 0$ by (2.9)) is defined by

$$W_\alpha{}^2 = \left\{ f(x) \in W_m{}^2; \left[\int_{E^n} \frac{\|D^k f(x+h) - D^k f(x)\|_2{}^2}{|h|^{2\beta+n}} dh \right]^{1/2} < \infty \right.$$
$$\left. \text{for every } |k| = m \right\}. \tag{5.1}$$

(See e.g. [20], [29], [27, p. 60], [32], and the papers cited there.) The following two spaces are special cases of the general Besov spaces. The space

$$\tilde{\mathfrak{B}}^{\alpha,2} = \left\{ f(x) \in L^2(E^n); \left[\int_{E^n} \frac{\|\Delta_h{}^r f(x)\|_2{}^2}{|h|^{2\alpha+n}} dh \right]^{1/2} < \infty \text{ for an } r > \alpha > 0 \right\} \tag{5.2}$$

is treated in [1]. Here

$$\Delta_h{}^r f(x) = \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f(x + \nu h).$$

Another type of Besov space (see [5, p. 90], [27, p. 78]) is given by

$$B_{2,2}^{\alpha} = \left\{ f(x) \in W_m{}^2; \left[\int_0^1 t^{-2\beta-1} \omega_{1+[\beta]}^2 \left(\frac{\partial^m f}{\partial x_j{}^m}, t\sigma^j \right) dt \right]^{1/2} < \infty \right.$$
$$\left. \text{for } j = 1, 2, \ldots, \infty \right\}. \tag{5.3}$$

Here the partial derivatives are taken in the distributional sense; $[\beta]$ is the greatest integer $\leqslant \beta$, and

$$\omega_r{}^2(\varphi, t\sigma^j) = \sup_{|\tau| \leqslant t} \|\Delta_{\tau\sigma^j}^r f(x)\|_2 \qquad (r = 1, 2).$$

Taibleson [32] defined the spaces

$$\Lambda(\alpha; 2, 2) = \left\{ f(x) \in L^2(E^n); \left[\int_0^\infty t^{-n} \left\| t^{r-\alpha} \frac{d^r}{dt^r} W_t{}^1(f; x) \right\|_2^2 dt \right]^{1/2} < \infty \right\}, \tag{5.4}$$

where α is positive, r the smallest integer greater than α, and $W_t{}^1$ the Cauchy–Poisson integral defined in (1.4) ($\kappa = 1$).

Collecting the results of [2, pp. 74, 82], [5, p. 109], [27, pp. 63, 79], and [32, p. 478] we have the following characterizations of $L_\alpha{}^2$ for $p = 2$:

(5.5) LEMMA. *Let $f(x) \in L^2(E^n)$, and $\alpha > 0$. Then the statements $f(x) \in L_\alpha{}^2$, $f(x) \in W_\alpha{}^2$, $f(x) \in \tilde{\mathfrak{B}}^{\alpha,2}$, $f(x) \in B_{2,2}^{(\alpha)}$, and $f(x) \in \Lambda(\alpha; 2, 2)$ are equivalent.*

Returning now to the general case $1 < p < \infty$, there are not as many characterizations for fractional α. For the sake of completeness we mention another equivalent condition of Stein [30].

(5.6) LEMMA. *For $0 < \alpha < 2$ and $f(x) \in L^p(E^n)$, $1 < p < \infty$, the following assertions are equivalent:*

(a) $f(x) \in L_\alpha^p$;

(b) $$\underset{\epsilon \to 0+}{\overset{(p)}{\text{l.i.m.}}} \frac{2^\alpha}{\pi^{n/2}} \frac{\Gamma((n+\alpha)/2)}{\Gamma(-\alpha/2)} \int_{|u| \geq \epsilon} \frac{f(x+u) - f(x)}{|u|^{n+\alpha}} du$$

exists.

For $0 < \alpha < 2$ and $2n/(n+2\alpha) < p < \infty$, these are, further, equivalent to

$$\left\{\int_{E^n} \frac{|f(x+u) - 2f(x) + f(x-u)|^2}{|u|^{n+2\alpha}} du\right\}^{1/2} < \infty.$$

Combining these equivalences with the results of Sections 3 and 4 we, furthermore, obtain for the particular cases $\alpha = 1, 2$ very concrete characterizations of the above abstract function classes.

Equivalences between ordinary and distributional derivatives have, as far as is known to the present author, as yet only been established by Aronszajn and Smith [2] and A. P. Calderón [16].

Finally we note that many further interesting characterizations have been obtained by Butzer and Berens [11] using the theory of intermediate spaces.

6. Applications to Saturation Theorems

In this final section we also use distributional methods in order to prove a saturation theorem. At the same time we obtain an affirmative answer to the question whether the class L_α^p, which may be regarded as an extension of V_α^p for $p > 2$ in view of Theorem (2.8), plays the role of the saturation class (Favard class) for the usual approximation processes of convolution type on $L^p(E^n)$ also for $p > 2$. As a representative example we consider in detail the generalized singular integral $W_t^\kappa(f;x)$ of Weierstrass defined by (1.4). This example is especially useful for our purpose since it contains a parameter $\kappa > 0$ which appears again in the saturation class V_κ^p.

As mentioned in the introduction, the saturation theorem for $W_t^\kappa(f;x)$ will indeed be established by a distributional method.[2] The saturation class will be shown to be the class L_κ^p, also for $p > 2$, and therefore all the previous

[2] This method, due to Dr. R. J. Nessel in the particular case $\alpha = 2$ (which is very elegant), was first presented by him in a seminar lecture held at Aachen on July 28, 1966. For a detailed discussion see also [22, p. 15 ff.].

characterizations can be applied here, particularly in the cases $\kappa = 1, 2$ when W_t^κ reduces to the singular integrals of Cauchy–Poisson and Gauss–Weierstrass, respectively. The saturation theorem reads

(6.1) THEOREM. *Let* $f(x) \in L^p(E^n)$, $1 < p < \infty$, *and* $\kappa > 0$. *Then*

(a) $\|W_t^\kappa(f; x) - f(x)\|_p = o(t) \quad (t \to 0+)$ *implies* $f(x) = 0$ a.e.;

(b) $\|W_t^\kappa(f; x) - f(x)\|_p = O(t) \quad (t \to 0+)$ *if and only if* $f(x) \in L_\kappa^p$.

For the proof we need the following

(6.2) *Fundamental Identity*. Let $f(x) \in L_\kappa^p$ for fixed p, κ with $\kappa > 0$, $1 < p < \infty$. Then

$$\frac{1}{t}\{f(x) - W_t^\kappa(f; x)\} = \frac{1}{t}\int_0^t W_\tau^\kappa(G_{-\kappa} * f * d\mu_\kappa^{(3)}; x)\, d\tau \tag{6.3}$$

holds for almost every x, where $\mu_\kappa^{(3)}$ is the bounded measure given by Lemma (2.7).

Proof. According to (2.5) we have $G_{-\kappa} * f = h(x) \in L^p$; thus both sides of (6.3) exist as elements of L^p. If $1 < p \leqslant 2$, then

$$[f(x) - W_t^\kappa(f; x)]^\wedge(v) = \{1 - e^{-t|v|^\kappa}\} f^\wedge(v),$$

and on the other hand

$$\left[\int_0^t W_\tau^\kappa(G_{-\kappa} * f * d\mu_\kappa^{(3)}; x)\, d\tau\right]^\wedge(v) = \int_0^t e^{-\tau|v|^\kappa}(1 + |v|^2)^{\kappa/2}[\mu_\kappa^{(3)}]^\vee(v)\, d\tau f^\wedge(v),$$

the interchange of the Fourier transform with the integral being justified as in [*26*, II, Lemma 4.1]. By Lemma (2.7) the assertion for $1 < p \leqslant 2$ follows by the identity

$$\frac{1}{t}\{1 - e^{-t|v|^\kappa}\} = \frac{1}{t}\int_0^t e^{-\tau|v|^\kappa}|v|^\kappa\, d\tau$$

and from the uniqueness of the Fourier transform on L^p.

If $2 < p < \infty$ and $G_{-\kappa} * f = h_f(x) \in L^p$, then we have for every $\varphi(x) \in S$ also $\varphi(x) \in L_\kappa^{p'} = V_\kappa^{p'}$ by Theorem (2.8) since $1 < p' < 2$. Thus, there exists a function $g_\varphi(x) \in L^{p'}$ with $|v|^\kappa \varphi^\wedge(v) = g^\wedge(v)$, which implies in view of Lemma (2.7)

$$|v|^\kappa \varphi^\wedge(v) = (1 + |v|^2)^{\kappa/2} \varphi^\wedge(v)\, \mu_\kappa^{(3)}(v) = [G_{-\kappa} * \varphi * d\mu_\kappa^{(3)}]^\wedge(v) = g_\varphi^\wedge(v).$$

Therefore $G_{-\kappa} * \varphi * d\mu_\kappa^{(3)} = g_\varphi(x) \in L^{p'}$ or $(\varphi * d\mu_\kappa^{(3)})(x) \in L_\kappa^{p'}$, and

$$(\varphi * d\mu_\kappa^{(3)})(x) = (G_\kappa * g_\varphi)(x).$$

With $\mu_\kappa^{(3)}(x) = \mu_\kappa^{(3)}(-x)$, which is clear from the definition of $\mu_\kappa^{(3)}$, we obtain for $\varphi(x) \in S$

$$\left\langle \int_0^t W_\tau^{\kappa}(h_f * d\mu^{(3)}; x)\, d\tau, \varphi(x) \right\rangle = \int_{E^n} h_f(x) \int_0^t W_\tau^{\kappa}(\varphi * d\mu_\kappa^{(3)}; x)\, d\tau\, dx$$
$$= \int_{E^n} h_f(x) \int_0^t W_\tau^{\kappa}(G_\kappa * g_\varphi; x)\, d\tau\, dx$$
$$= \int_{E^n} (G_\kappa * h_f)(x) \int_0^t W_\tau^{\kappa}(g_\varphi; x)\, d\tau\, dx.$$

So we obtain with $f(x) = (G_\kappa * h_f)(x)$

$$\left\langle t^{-1}\{W_t^{\kappa}(f; x) - f(x)\} - t^{-1} \int_0^t W_\tau^{\kappa}(G_{-\kappa} * f * d\mu_\kappa^{(3)}; x)\, d\tau, \varphi(x) \right\rangle$$
$$= \int_{E^n} f(x)\, t^{-1} \left\{ W_t^{\kappa}(\varphi; x) - \varphi(x) - \int_0^t W_\tau^{\kappa}(G_{-\kappa} * \varphi * d\mu_\kappa^{(3)}; x)\, d\tau \right\} dx. \qquad (6.4)$$

Since $1 < p' < 2$, we now obtain (6.3) by an application of the foregoing result to $\varphi(x) \in L_\kappa^{p'}$.

Proof of Theorem (6.1). If $f(x) \in L^p$, the fact that

$$\|W_t^{\kappa}(f; x) - f(x)\|_p = O(t) \qquad (t \to 0+)$$

holds, follows immediately by the fundamental identity (6.3).

Conversely, if this approximation holds, the weak compactness of $L^p(E^n)$ yields a sequence $\{t_r\}$ with $\lim_{r\to\infty} t_r = 0$ and a function $g(x) \in L^p$ such that in particular for every $\varphi(x) \in S$

$$\lim_{r\to\infty} \int_{E^n} t_r^{-1}\{f(x) - W_{t_r}^{\kappa}(f; x)\}\, \varphi(x)\, dx = \int_{E^n} g(x)\, \varphi(x)\, dx,$$

and thus also

$$\lim_{r\to\infty} \langle f(x), t_r^{-1}\{\varphi(x) - W_{t_r}^{\kappa}(\varphi; x)\}\rangle = \langle g(x), \varphi(x)\rangle \qquad (\varphi(x) \in S). \qquad (6.5)$$

On the other hand, we have for each $\varphi(x) \in S \subset L_\kappa^{p'}$

$$t^{-1}\{\varphi(x) - W_t^{\kappa}(\varphi; x)\} = t^{-1} \int_0^t W_\tau^{\kappa}(G_{-\kappa} * \varphi * d\mu_\kappa^{(3)}; x)\, d\tau$$

by (6.3). Furthermore

$$\underset{t\to 0+}{\overset{(p')}{\text{l.i.m.}}}\, t^{-1} \int_0^t W_\tau^{\kappa}(G_{-\kappa} * \varphi * d\mu_\kappa^{(3)}; x)\, d\tau = (G_{-\kappa} * \varphi * d\mu_\kappa^{(3)})(x),$$

and, in view of (6.5),

$$\langle f(x), (G_{-\kappa} * \varphi * d\mu_\kappa^{(3)})(x)\rangle = \lim_{r\to\infty} \left\langle f(x), t_r^{-1} \int_0^{t_r} W_\tau^{\kappa}(G_{-\kappa} * \varphi * d\mu_\kappa^{(3)}; x)\, d\tau \right\rangle$$
$$= \langle g(x), \varphi(x)\rangle.$$

Now, a similar argument as in the proof of (6.3) yields

$$\langle f(x), G_{-\kappa} * \varphi * d\mu_\kappa^{(3)}\rangle = \langle G_{-\kappa} * f * d\mu_\kappa^{(3)}, \varphi(x)\rangle,$$

thus we have for almost all x

$$(G_{-\kappa} * f * d\mu_\kappa^{(3)})(x) = g(x) \in L^p(E^n). \tag{6.6}$$

In order to show that $(G_{-\kappa} * f)(x) \in L^p$, we first suppose $1 < p \leqslant 2$. Then

$$[(G_{-\kappa} * f * d\mu_\kappa^{(3)})(x)]^\wedge(v) = (1 + |v|^2)^{\kappa/2} \mu_\kappa^{(3)\vee}(v) f^\wedge(v) = |v|^\kappa f^\wedge(v)$$

by Lemma (2.7), thus $|v|^\kappa f^\wedge(v) = g^\wedge(v)$ or $f(x) \in V_\kappa{}^p$. This, in turn, implies the assertion $f(x) \in L_\kappa{}^p$ or $(G_{-\kappa} * f)(x) \in L^p$ by Theorem (2.8).

If $2 < p < \infty$, we first consider $\varphi(x) \in S \subset L_\kappa^{p'}$ and apply the previous result to $\varphi(x)$. Then $(G_{-\kappa} * \varphi * d\mu_\kappa^{(3)})(x) \in L^{p'}$ and also the function $h_\varphi(x)$ defined by

$$h_\varphi(x) = (G_{-\kappa} * \varphi)(x) = (\varphi * d\mu_\kappa^{(1)})(x) + (G_{-\kappa} * \varphi * d\mu_\kappa^{(3)} * d\mu_\kappa^{(2)})(x)$$

belongs to $L^{p'}$ by Lemma (2.7).
In view of (6.6) we also have

$$h_f(x) = (f * d\mu_\kappa^{(1)*})(x) + (G_{-\kappa} * f * d\mu_\kappa^{(3)} * d\mu_\kappa^{(2)*})(x) \in L^p(E^n)$$

and it remains to show that $(G_{-\kappa} * f)(x) = h_f(x)$. Indeed,

$$\begin{aligned}\langle (G_{-\kappa} * f)(x), \varphi(x)\rangle &= \langle f(x), (G_{-\kappa} * \varphi)(x)\rangle \\ &= \langle f(x), (\varphi * d\mu_\kappa^{(1)})(x) + (G_{-\kappa} * \varphi * d\mu_\kappa^{(3)} * d\mu_\kappa^{(2)})(x)\rangle \\ &= \langle (f * d\mu_\kappa^{(1)*})(x) + (G_{-\kappa} * f * d\mu_\kappa^{(3)} * d\mu_\kappa^{(2)*})(x), \varphi(x)\rangle \\ &= \langle h_f(x), \varphi(x)\rangle,\end{aligned}$$

and thus $f(x) \in L_\kappa{}^p$.
If $\|W_t{}^\kappa(f;x) - f(x)\|_p = o(t)$, $t \to 0+$, then the same argument with $g(x) = 0$ gives $f(x) = 0$ a.e. This proves the Theorem.

We may also establish the corresponding results on Taylor differences for $W_t{}^\kappa(f;x)$ of the type treated by Butzer and Tillmann [*15*], Berens [*4*], and Butzer and Berens [*11*]. Indeed, using the same methods as above, only replacing (6.4) by

$$\begin{aligned}&W_t{}^\kappa(f;x) - \sum_{\nu=0}^{m-1} \frac{(-t)^\nu}{\nu!} (G_{-\nu\kappa} * f * d\mu_{\nu\kappa}^{(3)})(x) \\ &\quad = \frac{(-1)^m}{(m-1)!} \int_0^t (t-\tau)^{m-1} W_\tau{}^\kappa(G_{-m\kappa} * f * d\mu_{m\kappa}^{(3)}; x)\, d\tau \qquad (m = 1, 2, \ldots),\end{aligned}$$

we have

(6.7) THEOREM. *Let* $f(x) \in L^p(E^n)$, $1 < p < \infty$ *and* $m = 1, 2, \ldots$. *If* $f(x) \in L^p_{(m-1)\kappa}$, *then the following statements are equivalent:*

(a) $$\left\| W_t^\kappa(f;x) - f(x) - \sum_{\nu=1}^{m-1} \frac{(-t)^\nu}{\nu!} (G_{-\nu\kappa} * f * d\mu_{\nu\kappa}^{(3)})(x) \right\|_p = O(t^m)$$

$(t \to 0+)$;

(b) $f(x) \in L^p_{m\kappa}$.

In view of Theorems (3.2), (3.5), and (4.7), the class $L^p_{m\kappa}$ may be suitably characterized in case κ is an integer.

Similarly one can prove the saturation theorem for the singular integral of Bochner–Riesz defined by (cf. [*26*])

$$B_R^\alpha(f;x) = \frac{2^\alpha \Gamma(\alpha+1)}{(2\pi)^{n/2}} R^n \int_{E^n} f(x-u)(R|u|)^{-((n/2)+\alpha)} J_{(n/2)+\alpha}(R|u|)\,du,$$

where $f(x) \in L^p(E^n)$, $\alpha > (n-1)/2$, and $R > 0$. If $1 \leqslant p \leqslant 2$, we also have the representation

$$B_R^\alpha(f;x) = \frac{1}{(2\pi)^{n/2}} \int_{|v| \leqslant R} \left(1 - \frac{|v|^2}{R^2}\right)^\alpha f^\wedge(v)\, e^{i(v,x)}\,dv.$$

(6.8) THEOREM. *Let* $f(x) \in L^p(E^n)$, and $\alpha > (n-1)/2 + p$, $1 < p < \infty$. *Then*

(a) $\|B_R^\alpha(f;x) - f(x)\|_p = o(R^2)$, $R \to \infty$, *implies* $f(x) = 0$ a.e.

(b) $\|B_R^\alpha(f;x) - f(x)\|_p = O(R^2)$, $R \to \infty$, *if and only if* $f(x) \in L_2^p$.

Many further singular integrals can be treated by the same method, and the saturation class is always one of the spaces L_κ^p.

A paper by M. Kozima and G. Sunouchi: On the Approximation and Saturation by General Singular Integrals (in print) was drawn to the author's attention. It treats a functional method for establishing saturation theorems of the type considered in Section 6 of this paper and is similar to our method. For the origin of the method see also Footnote 2 of Section 6.

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